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A SURVEY OF THE COMPLEMENTED SUBSPACE PROBLEM

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Abstract. The complemented subspace problem asks, in general, which closed subspaces M of a Banach space X are complemented; i.e. there exists a closed subspace N of X such that $X = M \oplus N$? This problem is in the heart of the theory of Banach spaces and plays a key role in the development of the Banach space theory. Our aim is to investigate some new results on complemented subspaces, to present a history of the subject, and to introduce some open problems.

1. Introduction

The problem related to complemented subspaces are in the heart of the theory of Banach spaces. These are more than fifty years old and play a key role in the development of the Banach space theory. Our aim is to review of results on complemented subspaces, to present a history of the subject, and to introduce some open problems.

We start with simple observations concerning definition and properties of complemented subspaces. Some useful sources are [8, 17, 28].

Let X be a normed space, M, N be algebraically complemented subspaces of X (i.e. $M + N = X$ and $M \cap N = \{0\}$), $\pi : X \to \frac{X}{M}$ be the quotient map, $\phi : M \times N \to X$ be the natural isomorphism $(x, y) \mapsto x + y$ and $P : X \to M, P(x + y) = x, x \in M, y \in N$ be the projection of X on M along N. Then the following statements are equivalent:

(i) ϕ is a homeomorphism.

(ii) M and N are closed in X and $\pi|_N$ is a homeomorphism.

(iii) M and N are closed and $P : X \to M$ is a bounded projection.

The Subspaces M and N are called topologically complemented or simply complemented if each of the above equivalent statements holds. If N_1, N_2 are complemented subspaces of a closed subspace M, then N_1 and N_2 are isomorphic Banach spaces.

It is known that every finite dimensional subspace is complemented and every algebraic complement of a finite codimension subspace is topologically complemented.

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In a Banach space X, applying the closed graph theorem we can establish that two closed subspace are algebraically complemented if and only if they are complemented. Moreover, if M is a closed subspace of X, then M is complemented if and only if the following equivalent assertions hold:

(I) The quotient map $i : M \hookrightarrow X$ has a left inverse as a continuous operator .

(II) The natural projection $\pi : M \to \frac{X}{M}$ has a right inverse as a continuous operator.

l^∞ is complementary in every normed space X containing it isomorphically as a closed subspace [28]. Also, If $c_\circ$ is subspace of a separable Banach space X , then there is a bounded projection P of X onto $c_\circ$ of norm ≤ 2, cf. [44].

Suppose now that F is a retract of a Banach space X, i.e. F is a Banach subspace of X and there is a continuous linear map $\phi : X \to F$ such that for all $x \in F, \phi(x) = x$. Then $C_\circ(X - F) = \{f \in C(X) : f(x) = 0 \text{ for all } x \in F\}$ is complemented in $C(X)$. In fact, by defining $P : C(X) \to C(X)$ by $P(g) = g \circ \phi$, we have $P^2 = P, \|P(g)\| = \sup_{x\in X} |g(\phi(x))| \leq \|g\|$ and $KerP = \{g \in C(X)|g(\phi(x)) = 0 \text{ for all } x \in X\} = C_\circ(X - F)$.
Hence we may say that "complemented ideal' is the Gelfand dual of "retract closed subspace" (see [31]).

There are non-complemented closed subspaces. For example, let X be the disk algebra, i.e. the space of all analytic functions on $\{z \in \mathbf{C}; |z| < 1\}$ which are continuous on the closure of D. Then the subspace of $C(T)$ consisting of the restrictions of functions of X to $T = \{z \in \mathbf{C}; |z| = 1\}$ is not complemented in X (see [18]).

Throughout the paper $c_\circ, c, l_\infty, l_p$ denote the space of all complex sequences $\{x_n\}$ such that $\lim_{n\to\infty} x_n = 0$, $\{x_n\}$ is convergent, $\{x_n\}$ is bounded, and $\sum_{n=1}^{\infty} |x_n|^p < \infty$, respectively. In addition, L_p denotes the L_p-space over the Lebesgue interval $[0, 1]$. The reader is referred to [20, 26] for undefined terms and notation.

2. Complementary subspace problem and related results

This problem asks, in general, which closed subspaces of a Banach space are complemented?

In 1937, Murray [32] proved, for the first time, that $l_p, p \neq 2, p > 1$ has non-complemented subspace.

Phillips [38] proved that $c_\circ$ is non-complemented in l^∞. This significant fact has been refined, reproved or generalized by many mathematicians, cf. [37, 16, 42, 34].

Banach and Mazur showed that all subspaces in $C[0, 1]$ which are isometrically isomorphic to l_1 or $L^1[0, 1]$ are non-complemented, cf. [43, 1].

In 1960, Pelczynski [36] showed that complemented subspaces of l_1 are isomorphic to l_1. Köthe [22] generalized this result to the non-separable case.

In 1967, Lindenstrauss [25] proved that every infinite dimensional complemented subspace of l^∞ is isomorphic to l^∞. This also holds if l^∞ is replaced by l_p, $1 \leq p < \infty$, $c_\circ$ or c.

It is shown by Lindenstrauss [24] that if the Banach space X and its closed subspace Y are generated by weakly compact sets (in particular, if X is reflexive), then Y is complemented in X.

In 1971, Lindenstrauss and Tzafriri [26] proved that every infinite dimensional Banach space which is not isomorphic to a Hilbert space contains a closed non-complemented subspace.

Johnson and Lindenstrauss [19] proved the existence of a continuum of non-isomorphic separable $\mathcal{L}^1$-spaces. (An $\mathcal{L}^1$-space is a space X for which X^{**} is a complemented subspace of an L^1-space)

Classically known complemented subspaces of $L_p, 1 < p < \infty$ $, p \neq 2$ are $l_p, l_2, l_p \oplus l_2$ and L_p itself. In 1981, Bourgain, Rosenthal and Schechtman [3] proved that up to isomorphism, there exist uncountably many complemented subspaces of L_p.

It is shown that a complemented subspace M of l^*_∞ is isomorphic to l^*_∞ provided M is either w^*-closed or isomorphic to a bidual space, cf. [29].

Pisier [39] established that any complemented reflexive subspace of a C^*-algebra is necessarily linearly isomorphic to a Hilbert space.

In 1993, Gowers and Maurey [13] showed that there exists a Banach space X without non-trivial complemented subspaces.

If E is one of the spaces l_p, $(1 \leq p \leq \infty)$ or $c_\circ$, and X is a vector space complemented in E which contains a vector subspace Y complemented in X and isomorphic to E, then X is isomorphic to E. Moreover, each infinite dimensional vector subspace complemented in E is isomorphic to E. Conversely, if Y is a vector subspace of $E = l^2$ or $c_\circ$ which is isomorphic to E, then Y is complemented in E.

If X is an infinite dimensional vector subspace complemented in some space $C(S)$, then X contains a vector subspace isomorphic to $c_\circ$.

Randrianantoanina [40] showed that if X and Y are isometric subspaces of L_p $(p \neq 4, 6, ...)$, and X is complemented in L_p then so is Y. Moreover, the projection constant does not change. This number is defined to be $\inf\{\|T\| : T : L_p \to X$ is a bounded linear projection of L_p onto $X\}$.

The above theorem fails in the case $p \geq 4$ is an even integer, i.e. there exist pairs of isomorphic subspaces X and Y of L_p to itself so that X is complemented and Y is not.

3. Schroeder-Bernstein Problem

If two spaces are isomorphic to complemented subspaces of each other, are then they isomorphic?

There are negative solutions to this problem.(see [15, 14])

4. Basis and complemented subspaces

A Schauder basis for a Banach space X is a sequence $\{x_n\}$ in X with the property that every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} \alpha_n x_n; \alpha_n \in \mathbf{C}$

in which the sum is convergent in the norm topology, cf. [20]. For example, the trigonometrical system is a basis in each space $L^p[0,1], 1 < p < \infty$.

Pelczynski [36] showed that any Banach space with a basis is a complemented subspace of an isomorphically unique space.

In 1987, Szarek [45] showed that there is a complemented subspace without basis of a space with a basis and answered therefore to a problem of fifty years old.

5. Approximation property and complemented subspaces.

A Banach space X has the approximation property (AP) if for every $\epsilon > 0$ and each compact subset K of X there is a finite rank operator T in X such that for each $x \in K, \|Tx - x\| < \epsilon$. If there is a constant $C > 0$ such that for each such T, $\|T\| \leq C$, then X is said to have bounded approximation property (BAP), cf. [20]. For example, every Banach space with a basis has BAP.

Pelczynski [36] proved that every Banach space with the BAP can be complementably embedded in a Banach space with a basis.

6. Complemented minimal subspaces

A Banach space X is called minimal if every infinite dimensional subspace Y of X contains a subspace Z isomorphic to X. For example $c_\circ$ is minimal. If Z is also complemented then X is said to be complementary minimal. Casazza and Odell [5] showed that Tsirelson's space T (see [46, 12]) have no minimal subspaces.

Casazza, Johnson and Tzafriri [4] showed that the dual T^* of T is minimal but not complementary minimal.

7. Quasi-complemented subspaces

A closed subspace Y of a Banach space X is said to be quasi-complemented if there exists a closed subspace Z of X such that $Y \cap Z = \{0\}$ and $Y + Z$ is dense in X.

Then such a subspace Z is said to be a quasi-complement of Y. Those notions are first introduced by Murray [33].

Every closed subspace of l_∞ is quasi-complemented, cf. [42]. Also Mackey [27] proved that in a separable Banach space every subspace is quasi-complemented.

Rosenthal [41] showed that if X is a Banach space, Y is a closed subspace of X, Y^* is W^*-separable and the annihilator $Y^\perp$ of Y in X^* has an infinite dimensional reflexive subspace, then Y is quasi-complement in X.

8. Weakly complemented subspaces

A closed subspace of a Banach space X is called weakly complemented if the dual i^* of the natural embedding $i : M \hookrightarrow X$ has a right inverse as a bounded operator.

For example, $c_\circ$ is weakly complemented in l_∞, not complemented in l_∞ (see [47]).

If M is complemented in X with the corresponding projection P, then the adjoint of $id_X - P$ is a projection in $B(X)$ with the range $M^o = \{f \in X^*; f|_M = 0\}$. Hence M is weakly complemented in X.

9. Contractively complemented subspaces

As mentioned before, a closed subspace Y of a Banach space X is said to be complemented if it is the range of a bounded linear projection $P : X \to X$. If $\|P\| = 1$, Y is called a contractively complemented or 1-complemented subspace of X.

Let X be a Banach space with dim$X \geq 3$. Then X is isometrically isomorphic to a Hilbert space iff every subspace of X is the range of a projection of norm 1 (see [21, 2]).

In 1969, Zippin [48] proved that every separable infinite dimensional L_1-predual space (i.e a Banach space whose dual is isometric to $L_1(\mu)$ for some measure space (Ω, Σ, μ))) contains a contractively complemented subspace isomorphic to $c_\circ$.

Lindenstrauss and Lazar [23] proved that X contains a contractively complemented subspace isometric to some space $C(S)$ when X^* is non-separable.

Question. Let X be a Banach space and $T : X \to X$ be an isometry. Is the range of T is contractively complemented in X?
In Hilbert and $L^p, (1 \leq p < \infty)$ spaces, we have an affirmative answer. In case $C[0, 1]$, however, it may happen that the range of an isometry is not complemented, cf. [9].

Pisier [39] proved that if M is a von Neumann subalgebra of $B(H)$ which is complemented in $B(H)$ and isomorphic to $M \otimes M$, then M is contractively complemented.

10. Prime Banach spaces and complemented subspaces

A Banach space X is called prime if each infinite dimensional complemented subspace of X is isomorphic to X, cf. [26].

Pelczynski [36] proved that $c_\circ$ and l_p $(1 \leq p < \infty)$ are prime. Lindenstrauss [25] proved that l^∞ is also prime. Gowers and Maurey [13] constructed some new prime spaces.

11. Complemented subspaces of topological products and sums

Metafune and Moscatelli [30] proved that when X is one of the Banach spaces $l_p(1 \leq p \leq \infty)$ or $c_\circ$, then each infinite dimensional complemented subspace of X^N is isomorphic to one of the spaces $\omega, \omega \times X^N$ or X^N, where $\omega = K^N$ (K is the scaler field) and X^N is the product of countably many copies of X.

In [11], the authors obtained a complete description of the complemented subspace of the topological product l_∞^m where m is an arbitrary cardinal number.

Every complemented subspace of a product $V = \prod_{i \in I} X_i$ of Hilbert spaces is isomorphic to a product of Hilbert spaces (I is a set of arbitrary cardinal), cf. [10].

Ostraskii [35] showed that not all complemented subspaces of countable topological products of Banach spaces are isomorphic to topological products of Banach spaces.

Chigogidze [6] proved that complemented subspaces of a locally convex direct sum of arbitrary collection of Banach spaces are isomorphic to locally convex direct sum of complemented subspaces of countable subsums.

Chigogidze [7] proved that a complemented subspace of an uncountable topological product of Banach spaces is isomorphic to a topological product of complemented subspaces of countable subproducts and hence isomorphic to a topological product of Frechet spaces.

12. Some interesting problems

The following problems in this area arise:

1) Given a Banach space X, characterize the isomorphic types of its complemented subspaces.

2) Given a Banach space X, characterize the isomorphic types of such Banach space Z that every vector subspace of Z isomorphic to X is complemented in Z.

3) Is every complemented vector subspace of $C(S)$ isomorphic to some $C(S_1)$?

4) If a Banach space X is complemented in every Banach space containing it, is X isomorphism to some $C(S)$ over a Stone space S? (A space is Stonian if the closure of every open set is open)

5) Does every complemented subspace of a space with an unconditional basis have an unconditional basis? Recall that an unconditional basis for a Banach space is a basis $\{x_n\}$ such that every permutation of $\{x_n\}$ is also a basis or equivalently, the convergence of $\sum \alpha_n x_n$ implies the convergence of every rearrangement of the series, cf. [20].

6) If a von Neumann algebra is a complemented subspace of $B(H)$, is it then injective?

7) Are $l_p, 1 \leq p \leq \infty$ and $c_\circ$ the only prime Banach spaces with an unconditional basis? is still open.

Remark. Some pieces of information are taken from Internet-based resources without mentioning the URL's.

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