44th IMO 2003

A1. S is the set $\{1, 2, 3, \ldots, 1000000\}$. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $\{a + x_i | a \in A\}$ are all pairwise disjoint.

A2. Find all pairs (m, n) of positive integers such that $\frac{m^2}{2mn^2 - n^3 + 1}$ is a positive integer.

A3. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths Show that all the hexagon's angles are equal.

B1. $ABCD$ is cyclic. The feet of the perpendicular from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC iff $RP = RQ$.

B2. Given $n > 2$ and reals $x_1 \le x_2 \le \cdots \le x_n$, show that $(\sum_{i,j} |x_i - x_j|)^2 \le \frac{2}{3}(n^2 - 1) \sum_{i,j} (x_i - x_j)^2$. Show that we have equality iff the sequence is an arithmetic progression.

B3. Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.

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