The 8th Korean Mathematical Olympiad

[First round]

Morning session ($2\frac{1}{2}$ hours)

1. Consider a finitely many points in a plane such that, if we choose any three points A, B, C among them, the area of $\triangle ABC$ is always less than 1. Show that all of these finitely many points lie within the interior or on the boundary of a triangle with area less than 4.

2. For a given positive integer m, find all pairs (n, x, y) of positive integers such that m, n are relatively prime and $(x^2 + y^2)^m = (xy)^n$, where n, x, y can be represented by functions of m.

3. Let A, B, C be three points lying on a circle, and let P, Q, R be midpoints of arcs BC, CA, AB, respectively. AP, BQ, CR intersect BC, CA, AB at L, M, N, respectively. Show that
$$\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN} \geq 9.$$
For which triangle ABC does equality hold ?

4. A partition of a positive integer n is a sequence $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Each λ_i is called a summand. For example, $(4, 3, 1)$ is a partition of 8 whose summands are distinct. Show that, for a positive integer m with $n > \frac{1}{2}m(m+1)$, the number of all partitions of n into distinct m summands is equal to the number of all partitions of $n - \frac{1}{2}m(m+1)$ into r summands $(r \leq m)$.

fternoon session ($2\frac{1}{2}$ hours)

5. If we select at random three points on a given circle, find the probability that these three points lie on a semicircle.

6. Show that any positive integer $n(>1)$ can be expressed by a finite sum of numbers satisfying the following conditions:

   (i) they do not have factors exept 2 or 3.

   (ii) any two of them are neither a factor nor a multiple each other.

   That is, $n = \sum_{i=1}^{N} 2^{\alpha_i} 3^{\beta_i}$, where $\alpha_i, \beta_i \ (i = 1, 2, \cdots, N)$ nonnegative integers and $(\alpha_i - \alpha_j)(\beta_i - \beta_j) < 0$ whenever $i \neq j$.

7. Find all real valued functions f defined on real numbers except 0 such that

$$\frac{1}{x} f(-x) + f(\frac{1}{x}) = x, \qquad x \neq 0.$$

8. Two circles O_1, O_2 of radii $r_1, r_2 \ (r_1 < r_2)$, respectively, intersect at two points A and B. P is any point on a circle O_1. Lines PA, PB and a circle O_2 intersect at Q and R, respectively.

   (1) Express $y = QR$ in terms of r_1, r_2, and $\theta = \angle APB$.

   (2) Show that $y = 2r_2$ is a necessary and sufficient condition that a circle O_1 is orthogonal to a circle O_2.

[Final round]

First day session ($4\frac{1}{2}$ hours)

1. For any positive integer m, show that there exist integers a, b satisfying

$$|a| \leq m, \qquad |b| \leq m, \qquad 0 < a + b\sqrt{2} \leq \frac{1+\sqrt{2}}{m+2}.$$

2. Let A be the set of all non-negative integers. Find all functions $f : A \to A$ satisfying the following two conditions:

   (i) for any $m, n \in A$,

$$2f(m^2 + n^2) = \{f(m)\}^2 + \{f(n)\}^2$$

   (ii) for any $m, n \in A$ with $m \geq n$,

$$f(m^2) \geq f(n^2).$$

3. Let $\triangle ABC$ be an equilateral triangle of side length $1, D$ a point on BC, and let r_1, r_2, be inradii of triangles ABD, ADC, respectively. Express $r_1 r_2$ in terms of $p = BD$, and find the maximum of $r_1 r_2$.

Second day session ($4\frac{1}{2}$ hours)

4. Let O and R be the circumcenter and the circumradius of $\triangle ABC$, respectively, and let P be any point on the plane ABC. Let perpendiculars PA_1, PB_1, PC_1, be dropped to the three sides BC, CA, AB. Express $\frac{(\triangle A_1B_1C_1)}{(\triangle ABC)}$ in terms of R and $d = OP$, where $(\triangle ABC)$ is the area of $\triangle ABC$.

5. Let p be a prime number such that

   (i) p is the greatest common divisor of a and b;

   (ii) p^2 is a divisor of a. Prove that the polynomial $x^{n+2} + ax^{n+1} + bx^n + a + b$ cannot be decomposed into the product of two polynomials with integeral coefficients, whose degrees are greater tham one.

6. Let m, n be positive integers with $1 \leq n \leq m - 1$. A box is locked with several padlocks, all of which must be opened to open the box, and all of which have different keys. m people each have keys to some of the locks. No n people of them can open the box but any $n + 1$ people can open the box. Find the smallest number l of locks and then the number of keys for which this is possible.

The 8th Korean Mathemtical Olympiad
Solutions

[First round]

1. Consider a finitely many points in a plane such that, if we choose any three points A, B, C among them, the area of $\triangle ABC$ is always less than 1. Show that all of these finitely many points lie within the interior or on the boundary of a triangle with area less than 4.

Solution

We can take $\triangle ABC$ having the maximum area among triangles whose vertices are chosen from the given finitely many points. Then $(\triangle ABC) \leq 1$. Here $(\triangle ABC)$ denotes the area of $\triangle ABC$. Let $\triangle LMN$ be the triangle whose medial triangle is $\triangle ABC$. Then

$$(\triangle LMN) = 4(\triangle ABC) \leq 4.$$

We may prove that the given finitely many points lie within the interior or on the boundary of $\triangle LMN$. Suppose a point P lies on the exterior of $\triangle LMN$. We may assume that a point P locates at the opposite side of N with respect to a line ML. Then the length of a perpendicular of P dropping to AB is larger than that of C dropping to AB.

It follows that $(\triangle PAB) > (\triangle CAB) = (\triangle ABC)$ and P is never included in the given finitely many points. This completes the proof.

2. For a given positive integer m, find all pairs (n, x, y) of positive integers such that m, n are relatively prime and $(x^2 + y^2)^m = (xy)^n$, where n, x, y can be represented by functions of m.

Solution

If (n, x, y) is a solution of the given equation, then from $x^2 + y^2 \geq 2xy$ we have $(xy)^n = (x^2 + y^2)^m > (xy)^n$. It follows that $n > m$.

Let p be a common prime divisor of x and y, and let $p^a \| x, p^b \| y$. Here $p^a \| x$ means that $p^a | x$ but $p^{a+1} \nmid y$. Then $p^{(a+b)n} \| (xy)^n = (x^2 + y^2)^m$. Suppose $b > a$. Since $p^{2a} \| x^2, p^{2b} \| y^2$, we see that $p^{2a} \| x^2 + y^2$ and $p^{2am} \| (x^2 + y^2)^m$. It follows that $2am = (a + b)n > 2an$ and $m > n$. This is a contradiction. Similarly, $a > b$ gives a contradiction. It concludes that $a = b$ and $x = y$. Now we have $x^{2n} = (2x^2)^m = 2^m x^{2n}$ and $x^{2(n-m)} = 2^m$. Thus, $x = 2^t$ for some integer t. From

$2^{2(n-m)t} = 2^m$ we have $2t(n-m) = m$. Since $2nt = (2t+1)m$ and m, n are relatively prime, we obtain $m = 2t, n = 2t+1$. Answer. $(n, x, y) = (m+1, 2^{m/2}, 2^{m/2})$ (m: even)

3. Let A, B, C be three points lying on a circle, and let P, Q, R be midpoints of arcs BC, CA, AB, respectively. AP, BQ, CR intersect BC, CA, AB at L, M, N, respectively. Show that
$$\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN} \geq 9.$$
For which triangle ABC does equality hold ?

Solution

Since P is a midpoint of arc BC, we have $\angle BAP = \angle CAP = \frac{1}{2}\angle A$. It follows that $BL : CL = c : b$, and $BL = \frac{ca}{b+c}$, $CL = \frac{ba}{b+c}$. Now we have

$$AL \cdot PL = BL \cdot CL = \frac{a^2bc}{(b+c)^2} \tag{1}$$

Since $(\triangle ABL) + (\triangle ALC) = (\triangle ABC)$, we obtain $\frac{1}{2}cAL\sin\frac{A}{2} = \frac{1}{2}bAL\sin\frac{A}{2} = \frac{1}{2}bc\sin A$ and
$$AL = \frac{bc}{b+c}\frac{\sin A}{\sin\frac{A}{2}} = \frac{2bc}{b+c}\cos\frac{A}{2}.$$
Form (1) we have
$$PL = \frac{a^2bc}{(b+c)^2}\cdot\frac{b+c}{2bc}\cdot\frac{1}{\cos\frac{A}{2}} = \frac{a^2}{2(b+c)}\cdot\frac{1}{\cos\frac{A}{2}} \quad \text{and}$$
$$\frac{AL}{PL} = \frac{4bc}{a^2}\cos^2\frac{A}{2} = \frac{2bc}{a^2}(1+\cos A) = \frac{2bc}{a^2}\Big(1 + \frac{b^2+c^2-a^2}{2bc}\Big) = \Big(\frac{b+c}{a}\Big)^2 - 1.$$
Similarly,
$$\frac{BM}{QM} = \Big(\frac{c+a}{b}\Big)^2 - 1 \quad \text{and} \quad \frac{CN}{RN} = \Big(\frac{a+b}{c}\Big)^2 - 1.$$
Finally,
$$\begin{aligned}
&\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN} = \Big(\frac{b+c}{a}\Big)^2 + \Big(\frac{c+a}{b}\Big)^2 + \Big(\frac{a+b}{c}\Big)^2 - 3 \\
= &\Big(\frac{b}{a} - \frac{a}{b}\Big)^2 + \Big(\frac{c}{b} - \frac{b}{c}\Big)^2 + \Big(\frac{a}{c} - \frac{c}{a}\Big)^2 + 2abc\Big(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\Big) + 3 \\
= &\Big(\frac{b}{a} - \frac{a}{b}\Big)^2 + \Big(\frac{c}{b} - \frac{b}{c}\Big)^2 + \Big(\frac{a}{c} - \frac{c}{a}\Big)^2 \\
&+ (ab+bc+ca)\Big\{\Big(\frac{1}{a} - \frac{1}{b}\Big)^2 + \Big(\frac{1}{b} - \frac{1}{c}\Big)^2 + \Big(\frac{1}{c} - \frac{1}{a}\Big)^2\Big\} + 9 \geq 9,
\end{aligned}$$

equality holds for $a = b = c$.

4. A partition of a positive integer n is a sequence $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Each λ_i is called a summand. For example, $(4, 3, 1)$ is a partition of 8 whose summands are distinct. Show that, for a positive integer m with $n > \frac{1}{2}m(m+1)$, the number of all partitions of n into distinct m summands is equal to the number of all partitions of $n - \frac{1}{2}m(m+1)$ into r summands $(r \leq m)$.

Solution

Since a partition of n into distinct m summands is of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n,\ \lambda_1 > \lambda_2 > \cdots > \lambda_m \geq 1$, we have $\lambda_m \geq 1,\ \lambda_{m-1} \geq 2,\ \lambda_{m-2} \geq 3, \cdots, \lambda_1 \geq m$. It follows that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m \geq 1 + 2 + \cdots + m = m(m+1)/2$. This partition can be represented geometrically. Consider the array of points having λ_1 points in the top row, λ_2 in next row, and so on down to λ_m in the bottom row.

$$\begin{array}{lll} \lambda_1 & \bullet\ \bullet\ \bullet\ \bullet \cdots \bullet \cdots \bullet & \text{more than } m \text{ points} \\ \lambda_2 & \bullet\ \bullet\ \bullet\ \bullet \cdots \bullet & \text{more than } m-1 \text{ points} \\ \cdots & \cdots\cdots\cdots & \\ \lambda_{m-1} & \bullet\ \bullet\ \cdots & \text{more than 2 points} \\ \lambda_m & \bullet\ \cdots & \text{more than 1 point} \end{array} \qquad (1)$$

In the first row, we divide λ_1 points into m points and $\lambda_1 - m$ points, and in the second row, we divide λ_2 points into $m-1$ points and $\lambda_2 - (m-1)$ points, etc.

$$\begin{array}{rclllll} \lambda_1 & : & \bullet\ \bullet\ \bullet \cdots \bullet\ \bullet & m \text{ points} & \cdots\cdots & (\lambda_1 - m) \text{ points} \\ \lambda_2 & : & \bullet\ \bullet\ \bullet \cdots \bullet & m-1 \text{ points} & \cdots\cdots & (\lambda_2 - (m-1)) \text{ points} \\ & & \cdots\cdots & & \cdots\cdots & \\ \lambda_{m-1} & : & \bullet\ \bullet & 2 \text{ points} & \cdots\cdots & (\lambda_{m-1} - 2) \text{ points} \\ \lambda_m & : & \bullet & 1 \text{ points} & \cdots\cdots & (\lambda_m - 1) \text{ points} \\ \hline & & & \text{total: } \dfrac{m(m+1)}{2} \text{ points} & & \text{total: } n - \dfrac{1}{2}m(m+1) \text{ points} \end{array} \qquad (2)$$

Since $\lambda_i - (m+1-i) \geq 0$ and $\sum_{i=1}^{m} \{\lambda_i - (m+1-i)\} = \sum_{i=1}^{n} \lambda_i - \frac{m(m+1)}{2} = n - \frac{m(m+1)}{2}$, the right part of the array of points represents a partition of $n - \frac{1}{2}m(m+1)$ into r summands $(r \leq n)$. This completes the proof.

5. If we select at random three points on a given circle, find the probability that these three points lie on a semicircle.

Solution

Let A, B, C be selected points on a given circle, and let $\angle AOB = x, \angle BOC = y, \angle COA = z$. Then

$$(1) \qquad x + y + z = 2\pi \qquad x \geq 0,\ y \geq 0,\ z \geq 0.$$

Three points A, B, C lie on a semicircle if and only if one of $x+y \leq z, y+z \leq x$ or $z+x \leq y$ holds, that is,

$$(2) \qquad x \geq \pi,\ y \geq \pi \ \text{ or } \ z \geq \pi.$$

In space, the region of (1) is the union of the interior and the boundary of $\triangle ABC$ and the region of (2) is the union of the interiors and the boundaries of $\triangle A'B'C,\ \triangle AB'C'$ and $\triangle BC'A'$.
It follows that the required probability is $\frac{3}{4}$. Answer. $\frac{3}{4}$.

cf. The required probability is equal to the probability that $\triangle ABC$ becomes obtuse.

6. Show that any positive integer $n(>1)$ can be expressed by a finite sum of numbers satisfying the following conditions:

   (i) they do not have factors exept 2 or 3.

   (ii) any two of them are neither a factor nor a multiple each other.

   That is, $n = \sum_{i=1}^{N} 2^{\alpha_i}3^{\beta_i}$, where $\alpha_i, \beta_i \ (i = 1, 2, \cdots, N)$ nonnegative integers and $(\alpha_i - \alpha_j)(\beta_i - \beta_j) < 0$ whenever $i \neq j$.

Solution

Note that $2 = 2^1, 3 = 3^1, 4 = 2^2, 5 = 2^1+3^1, 6 = 2^1 \cdot 3^1, 7 = 2^2+3^1, 8 = 2^3, 9 = 3^2, 10 = 2^2+2^1 \cdot 3^1$, etc. So if $n \leq 10$, then the required representation is possible.

We use induction on n.

Suppose that, for every $m(\leq n)$, the required representation of m is possible.

i) n is odd;

Since $n+1$ is even and $\frac{n+1}{2} < n$, we have $\frac{n+1}{2} = \sum_{i=1}^{N} 2^{\alpha_i} \cdot 3^{\beta_i}$, where $(\alpha_i - \alpha_j)(\beta_i - \beta_j) < 0$ whenever $i \neq j$. Then $n+1 = \sum_{i=1}^{N} 2^{\alpha_i+1} \cdot 3^{\beta_i}$ and $\{(\alpha_i+1)-(\alpha_j+1)\}(\beta_i-\beta_j) = (\alpha_i-\alpha_j)(\beta_i-\beta_j) < 0$ whenever $i \neq j$.

ii) n is even ;

If $n+1 = 3^k$, we have done.

If $n+1 \neq 3^k$, we can take a positive integer k such that $3^k < n+1 < 3^{k+1}$.

Since 3^k is odd, $n+1-3^k$ is even, and $(n+1-3^n)/2 < n$, we have $(n+1-3^k)/2 = \sum_{i=1}^{N} 2^{\alpha_i} \cdot 3^{\beta_i}$, where $(\alpha_i - \alpha_j)(\beta_i - \beta_j) < 0$ whenever $i \neq j$.

Then $n+1 = 3^k + \sum_{i=1}^{N} 2^{\alpha_i+1} \cdot 3^{\beta_i}$. Since $\alpha_i + 1 > 0, \beta_i < k \quad (i = 1, 2, \cdots, N)$, we have 3^k and $2^{\alpha_i+1} \cdot 3^{\beta_i} \quad (i = 1, 2, \cdots, N)$ are neither a factor nor a multiple each other. From i) and ii) we know that the required representation of $n+1$ is possible. This completes the proof.

7. Find all real valued functions f defined on real numbers except 0 such that

$$\frac{1}{x} f(-x) + f(\frac{1}{x}) = x, \qquad x \neq 0.$$

Solution

Letting $x = -y$ in the given equation, we have

$$-\frac{1}{y} f(y) + f(-\frac{1}{y}) = -y \tag{1}$$

and letting $x = 1/y$, we have

$$y f(-\frac{1}{y}) + f(y) = \frac{1}{y}. \tag{2}$$

Eliminating $f(-\frac{1}{y})$ in (1) and (2), we have

$$f(y) = \frac{1}{2}(y^2 + \frac{1}{y}).$$

Conversely, if $f(x) = \frac{1}{2}(x^2 + 1/x)$, then

$$\frac{1}{x} f(-x) + f(\frac{1}{x}) = \frac{1}{x} \cdot \frac{-x^3+1}{-2x} + \frac{\frac{1}{x^3}+1}{2 \cdot \frac{1}{x}} = x.$$

Answer. $\quad f(x) = \frac{1}{2}(x^2 + 1/x), \qquad x \neq 0.$

8. Two circles O_1, O_2 of radii $r_1, r_2 \ (r_1 < r_2)$, respectively, intersect at two points A and B. P is any point on a circle O_1. Lines PA, PB and a circle O_2 intersect at Q and R, respectively.

(1) Express $y = QR$ in terms of r_1, r_2, and $\theta = \angle APB$.

(2) Show that $y = 2r_2$ is a necessary and sufficient condition that a circle O_1 is orthogonal to a circle O_2.

Solution

(1) Case I. P lies in the exterior of a circle O_2;

Let $\angle PAB = \alpha$ and $AB = a$. Then $a = 2r_1 \sin\theta$,

$$\frac{PR}{PA} = \frac{PQ}{PB} = \frac{QR}{AB} = \frac{y}{a}, \quad \frac{BQ}{\sin\alpha} = 2r_2, \quad \text{and}$$
$$\frac{PB}{\sin\alpha} = \frac{a}{\sin\theta} = \frac{PA}{\sin(\theta+\alpha)} = 2r_1.$$

In $\triangle PBQ$, we have

$$BQ^2 = PB^2 + PQ^2 - 2PB\cdot PQ\cos(\angle BPQ) = PB^2\Big[1 + \frac{y^2}{a^2} - 2\frac{y}{a}\cos(\angle BPQ)\Big].$$

It follows that

$$4r_2^2\sin^2\alpha = 4r_1^2\sin^2\alpha[\{\frac{y}{a} - \cos(\angle BPQ)\}^2 + \sin^2(\angle BPQ)] \quad \text{and}$$
$$y = a\cos\theta + \frac{a}{n}\sqrt{r_2^2 - r_1^2\sin^2\theta} = 2\sin\theta\Big\{r_1\cos\theta + \sqrt{r_2^2 - r_1^2\sin^2\theta}\Big\}.$$

Case II. P lies in the interior of a circle O_2;

Similarly,

$$y = 2\sin\theta\Big\{r_1\cos(\pi-\theta) + \sqrt{r_2^2 - r_1^2\sin^2\theta}\Big\},$$

where

$$\angle BPQ = \pi - \angle APB = \pi - \theta.$$

(2) If a circle O_1 is orthogonal to a circle O_2, then $r_2 = r_1\tan\theta$ (Case I); $r_2 = r_1\tan(\pi-\theta)$ (Case II). It follows that $y = 2r_2$. Conversely, if $y = 2r_2$, then $r_2 = r_1\tan\theta$ (Case I); $r_2 = r_1\tan(\pi-\theta)$ (Case II). This completes the proof. Answer of (1);

$$y = 2\sin\theta\{r_1\cos\theta + \sqrt{r_2^2 - r_1^2\sin^2\theta}\}$$
(P lies in the exterior of a circle O_2),
$$y = 2\sin\theta\{r_1\cos(\pi-\theta) + \sqrt{r_2^2 - r_1^2\sin^2\theta}\}$$
(P lies in the interior of a circle O_2).

The 8th Korean Mathemtical Olympiad

Solutions

[Final round]

1. For any positive integer m, show that there exist integers a, b satisfying

$$|a| \le m, \qquad |b| \le m, \qquad 0 < a + b\sqrt{2} \le \frac{1+\sqrt{2}}{m+2}.$$

Solution

Let $f(x, y) = x + y\sqrt{2}$ and $S = \{f(a, b) | a, b \text{ integers with } 0 \le a \le m \text{ and } 0 \le b \le m\}$. If $f(a, b) = f(a', b')$, then $a + b\sqrt{2} = a' + b'\sqrt{2}$ and $a = a', b = b'$. It follows that the number of elements in S is $(m+1)^2$. The maximum of S is $m + m\sqrt{2} = (1+\sqrt{2})m$, and so if $f(a, b) \in S$, then $0 \le f(a, b) \le (1+\sqrt{2})m$. Devide the interval $[0, (1+\sqrt{2})m]$ into equal $m^2 + 2m$ small intervals. Then the lengths of small intervals are $\frac{(1+\sqrt{2})m}{m^2+2m} = \frac{1+\sqrt{2}}{m+2}$. Since the number of elements in S is $(m+1)^2 = (m^2+2m)+1$, by peogion hole principle there exist $f(a_1, b_1), f(a_2, b_2) \in S$ such that $f(a_1, b_1), f(a_2, b_2)$ lie on the same small interval. We may assume that $f(a_1, b_1) > f(a_2, b_2)$. Then $0 < f(a_1, b_1) - f(a_2, b_2) < \frac{1+\sqrt{2}}{m+2}$.

Let $a = a_1 - a_2$ and $b = b_1 - b_2$. Then

$$f(a_1, b_1) - f(a_2, b_2) = f(a_1 - a_2, b_1 - b_2) = f(a, b)$$

$0 < f(a, b) \le \frac{1+\sqrt{2}}{m+2}$, and $|a| \le m, |b| \le m$. This completes the proof.

2. Let A be the set of all non-negative integers. Find all functions $f : A \to A$ satisfying the following two conditions:

(i) for any $m, n \in A$,

$$2f(m^2 + n^2) = \{f(m)\}^2 + \{f(n)\}^2$$

(ii) for any $m, n \in A$ with $m \ge n$,

$$f(m^2) \ge f(n^2).$$

Solution

From (i) we have

$$2f(m^2) = \{f(m)\}^2 + \{f(0)\}^2$$
$$2f(n^2) = \{f(n)\}^2 + \{f(0)\}^2$$

and substrating two equations, we obtain

$$\{f(m)\}^2 - \{f(n)\}^2 = 2\{f(m^2) - f(n^2)\}.$$

It follows that if $m \geq n$, then by (ii) we see that

$$f(m) \geq f(n). \tag{1}$$

Meauwhile, if $m = n = 0$, then we have $2f(0) = \{f(0)\}^2$ and, $f(0) = 0$ or $f(0) = 1$.

Case I. $f(0) = 1$.

From $2f(m^2) = \{f(m)\}^2 + 1$, we have

$$f(2^{2^n}) = \frac{1}{2}[\{f(2^{2^{n-1}})\}^2 + 1]. \tag{2}$$

From $2f(1) = \{f(1)\}^2 + 1$ we have $\{f(1) - 1\}^2 = 0$ and $f(1) = 1$. From (2) we have

$$f(2) = 1, \quad f(2^2) = 1, \cdots, \text{ etc.}$$

In general, $f(2) = f(2^2) = \cdots = f(2^{2^n}) = 1 \ \ (n \geq 0)$.

For any positive integer m, there exists an integer n such that $2^{2^{n-1}} \leq m < 2^{2^n}$.

From (1) we have $1 = f(2^{2^{n-1}}) \leq f(m) \leq f(2^{2^n}) = 1$ and finally $f(m) = 1$.

Case II. $f(0) = 0$.

From $2f(m^2) = \{f(m)\}^2$ we have

$$\frac{f(m^2)}{2} = \{\frac{f(m)}{2}\}^2. \tag{3}$$

Meanwhile, $f(2) = \{f(1)\}^2$. It follows that

$$\frac{f(2^{2^n})}{2} = \left\{\frac{f(2^{2^{n-1}})}{2}\right\}^2 = \left\{\frac{f(2^{2^{n-2}})}{2}\right\}^{2^2} = \cdots = \left\{\frac{f(2)}{2}\right\}^{2^n} = \frac{\{f(1)\}^{2^{n+1}}}{2^{2^n}} \tag{4}$$

From $2f(1) = \{f(1)\}^2$ we have $f(1) = 0$ or $f(1) = 2$.

(i) $f(1) = 0$.

From (4) we have $f(2^{2^n}) = 0 \ \ (n \geq 0)$. Similarly, $f(n) = 0 \ \ (n \geq 0)$.

(ii) $f(1) = 2$.

From (4) we have $f(2^{2^n}) = 2 \cdot 2^{2^n} \ \ (n \geq 0)$. By (3) we have $f(m)$ is even.

$$\begin{aligned} \{f(m+1)\}^2 = \quad & 2f((m+1)^2) \geq 2f(m^2+1) = \{f(m)\}^2 + \{f(1)\}^2 \\ > \quad & \{f(m)\}^2 \\ & \text{implies that } \ f(m+1) > f(m). \end{aligned}$$

It follows that $f(m+1) - f(m) - 2 \geq 0$, and $\sum_{m=0}^{2^{2^{n-1}}} \{f(m+1) - f(m) - 2\} = f(2^{2^n}) - f(0) - 2 \cdot 2^{2^n} = 0$. Now we have $f(m+1) - f(m) - 2 = 0$ for all $m = 0, 1, \cdots 2^{2^n} - 1$. Since n is arbitrary, we obtain

$f(m+1) = f(m) + 2$ for all $m = 0, 1, 2, \cdots$. It follows that $f(m) = 2m$. Answer. $f(n) \equiv 1$; $f(n) \equiv 0$; $f(n) \equiv 2n$.

3. Let $\triangle ABC$ be an equilateral triangle of side length $1, D$ a point on BC, and let r_1, r_2, be inradii of triangles ABD, ADC, respectively. Express $r_1 r_2$ in terms of $p = BD$, and find the maximum of $r_1 r_2$.

Solution

Let $l = AD$. In $\triangle ABD$, we have
$l^2 = 1 + p^2 - 2p\cos 60^\circ = p^2 - p + 1$.
Since the area of $\triangle ABC$ is $\sqrt{3}/4$,
in $\triangle ABD$ and $\triangle ADC$, we obtain

$$r_1 \frac{1+p+l}{2} = \frac{\sqrt{3}}{4}p, \quad r_2 \frac{2-p+l}{2} = \frac{\sqrt{3}}{4}(1-p).$$

Now we have

$$\begin{aligned} r_1 r_2 &= \frac{3}{4}\frac{p(1-p)}{(1+p+l)(2-p+l)} = \frac{3}{4}\frac{1}{(1+p)(2-p)+l^2+3l} \\ &= \frac{3}{4}\frac{p(1-p)}{2-p-p^2+p^2-p+1+3l} = \frac{1}{4}\frac{p(1-p)}{1+l} \\ &= \frac{1}{4}\frac{p(1-p)(1-l)}{4(1-l^2)} = \frac{1}{4}\frac{p(1-p)(1-l)}{1-p^2+p-1} = \frac{1}{4}(1-l) \\ &= \frac{1}{4}\left(1-\sqrt{p^2-p+1}\right) = \frac{1}{4}\left(1-\sqrt{(p-1/2)^2+3/4}\right) \\ &\leq \frac{1}{4}\left(1-\frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{8}. \end{aligned}$$

Answer. $r_1 r_2 = \frac{1}{4}(1-\sqrt{p^2-p+1})$; the maximum of $r_1 r_2$ is $\frac{2-\sqrt{3}}{8}$ when $p = \frac{1}{2}$.

4. Let O and R be the circumcenter and the circumradius of $\triangle ABC$, respectively, and let P be any point on the plane ABC. Let perpenticulars PA_1, PB_1, PC_1, be dropped to the three sides BC, CA, AB. Express $\frac{(\triangle A_1B_1C_1)}{(\triangle ABC)}$ in terms of R and $d = OP$, where $(\triangle ABC)$ is the area of $\triangle ABC$.

Solution

A quadrilateral PA_1CB_1 is inscribed in a circle of a diameter CP. It follows that $A_1B_1 = PC \cdot \sin C$. Similarly, in a quadrilateral PB_1AC_1 we have $B_1C_1 = AP \cdot \sin A$.

Let D be the intersection point between
a line CP and the circumcircle of $\triangle ABC$.

Since $\angle PB_1C_1 = \angle PCA_1 = \angle DAB$ and
$\angle PB_1C_1 = \angle PAC$, we see that
$\angle A_1B_1C_1 = \angle PB_1A_1 + \angle PB_1C_1 = \angle DAB + \angle PAC_1 = \angle PAD$.
From $\triangle PAD$ we have $AP/\sin B = DP/\sin(\angle PAD) = DP/\sin \angle A_1B_1C_1$ because $\angle PDA = \angle B$.

It follow that $AP \cdot \sin(\angle A_1B_1C_1) = DP \sin B$. Now we have

$$\begin{aligned}(\triangle A_1B_1C_1) &= \frac{1}{2}A_1B_1 \cdot B_1C_1 \cdot \sin(\angle A_1B_1C_1)\\ &= \frac{1}{2}(PC \cdot \sin C) \cdot (AP \cdot \sin A) \cdot \sin(\angle A_1B_1C_1)\\ &= \frac{1}{2}PC \cdot \sin C \cdot \sin A \cdot DP \sin B\\ &= \frac{1}{2}PC \cdot DP \cdot \sin A \sin B \sin C.\end{aligned}$$

Since $(\triangle ABC) = 2R^2 \sin A \sin B \sin C$, we obtain

$$\frac{(\triangle A_1B_1C_1)}{(\triangle ABC)} = \frac{1}{4R^2}\ PC \cdot DP.$$

If P lies in the interior of a circle O, then $PC \cdot DP = R^2 - d^2$, and if P lies in the exterior of a circle O, then $PC \cdot DP = d^2 - R^2$. Finally, if P lies on the circle O, then A_1, B_1, C_1 are collinear (Simson's line) and $(\triangle A_1B_1C_1) = 0$. In summary, we obtain

$$\frac{(\triangle A_1B_1C_1)}{(\triangle ABC)} = \frac{|R^2 - d^2|}{4R^2}.$$

5. Let p be a prime number such that

   (i) p is the greatest common divisor of a and b;

   (ii) p^2 is a divisor of a. Prove that the polynomial $x^{n+2} + ax^{n+1} + bx^n + a + b$ cannot be decomposed into the product of two polynomials with integeral coefficients, whose degrees are greater tham one.

Solution

(Eisenstein Theorem) Let $f(x) = c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$ be a polynomial with integral coefficients. If there exists a prime number p so that $p \nmid c_0, p|c_i\ \ (i = 1, 2, \cdots, n)$, and $p^2 \nmid c_n$, then $f(x)$ cannot be decomposed into the product of two polynomials with of lower degrees with integral coefficients.

(Proof) Let

$$\Big(\sum_{i=0}^{n+1} a_i x^i\Big)\Big(\sum_{j=0}^{n+1} b_j x^j\Big) = c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n.$$

From $a_0b_0 = c_n, p^2 \nmid c_n$ we have $p|a_0, p \nmid b_0$ or $p \nmid a_0, p|b_0$.

Without loss of generality, we may assume that $p|a_0,\ p \nmid b_0$.

Comparing the coefficients of x, we have $a_0b_1 + a_1b_0 = c_1$. It follows that $p|a_1b_0$ and $p|a_1$. Suppose $p|a_0, p|a_1, \cdots, p|a_{k-1}$. Then comparing the coefficients of x^k, we have $p|\sum_{i=0}^{k} a_ib_{k-i}$ and $p|a_kb_0$. It follows that $p|a_k$. Now we have $p|a_i(i = 0, 1, 2, \cdots, n)$. If $\deg(\sum_{i=0}^{n+1} a_i) = s$, then $\deg(\sum_{i=0}^{n+1} b_ix^i) = n - s$. Now we have $a_sb_{n-s} = c_0$ and $p|c_0$. This is a contradiction, completing proof.

Since $p \nmid 1, p|a, p|b, p^2 \nmid (a + b)$, by Eisenstein Theorem we see that the required proposition holds.

6. Let m, n be positive integers with $1 \leq n \leq m - 1$. A box is locked with several padlocks, all of which must be opened to open the box, and all of which have different keys. m people each have keys to some of the locks. No n people of them can open the box but any $n + 1$ people can open the box. Find the smallest number l of locks and then the number of keys for which this is possible.

Solution

Let m people be denoted by $A_1, A_2, \cdots, A_m$ and write on each lock the symbols of people who do not hold a key for that lock. Since any $n + 1$ people can open the box, no lock has over $n + 1$ symbols on it. For any n people, there is at least one lock with the corresponding n symbols. Thus the smallest number l of locks is ${}_mC_n$ and any people must have the same number of keys ($k = {}_{m-1}C_n$).